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ON CERTAIN THEOREMS OF JACOBI RELATING TO CANONICAL EQUATIONS.

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Jacobi in his *Vorlesungen über Dynamik* first proves the particular theorem contained in Art. 5 of the present paper, and afterwards derives therefrom that given in Art. 6. In what follows it will be seen that each of these is made a special case of the more general theorem given in Art. 1.

1. If we have given the partial differential equations

$$\frac{\partial S}{\partial t} + H = 0, \quad (a)$$

$$y_1 = \frac{\partial S}{\partial x_1}, \quad y_2 = \frac{\partial S}{\partial x_2}, \quad \dots, \quad y_{2n} = \frac{\partial S}{\partial x_{2n}}, \quad (b)$$

in which H is a function of t and of the x 's and y 's, but S is a function of t and the x 's only; and if there exist the so-called canonical relations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}; \quad (i = 1, 2, \dots, n) \quad (c)$$

then will similar canonical relations exist between the remaining variables; that is to say, we shall have

$$\frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial y_{n+i}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}. \quad (i = 1, 2, \dots, n) \quad (d)$$

Differentiating each of the y 's totally with reference to t , we have

$$\left. \begin{aligned} \frac{\partial y_1}{\partial t} + \sum_1^{2n} \frac{\partial y_1}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_1}{dt} &= 0, \\ \frac{\partial y_2}{\partial t} + \sum_1^{2n} \frac{\partial y_2}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_2}{dt} &= 0, \\ \dots & \dots \dots \dots \dots \dots \dots \\ \frac{\partial y_{2n}}{\partial t} + \sum_1^{2n} \frac{\partial y_{2n}}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_{2n}}{dt} &= 0. \end{aligned} \right\} \quad (1)$$

Also, since

$$\frac{\partial}{\partial x_i} \frac{\partial S}{\partial t} = \frac{\partial}{\partial t} \frac{\partial S}{\partial x_i} = \frac{\partial y_i}{\partial t},$$

if we differentiate equations (a) partially with reference to x_1, x_2, \dots, x_{2n} , we shall have

$$\left. \begin{aligned} \frac{\partial y_1}{\partial t} + \frac{\partial H}{\partial x_1} + \sum_1^{2n} \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_1} &= 0, \\ \frac{\partial y_2}{\partial t} + \frac{\partial H}{\partial x_2} + \sum_1^{2n} \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_2} &= 0, \\ \dots & \dots \dots \dots \dots \dots \dots \\ \frac{\partial y_{2n}}{\partial t} + \frac{\partial H}{\partial x_{2n}} + \sum_1^{2n} \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_{2n}} &= 0. \end{aligned} \right\} \quad (2)$$

Since

$$\frac{\partial}{\partial x_j} \frac{\partial S}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\partial S}{\partial x_j},$$

we may write

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i} = a_{i,j};$$

whence by reason of equations (c), we may also write

$$\frac{\partial y_1}{\partial t} + \sum_1^n \frac{\partial y_1}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_1}{dt} = \frac{\partial y_1}{\partial t} + \frac{\partial H}{\partial x_1} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_1} = \lambda_1,$$

$$\frac{\partial y_2}{\partial t} + \sum_1^n \frac{\partial y_2}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_2}{dt} = \frac{\partial y_2}{\partial t} + \frac{\partial H}{\partial x_2} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_2} = \lambda_2,$$

$$\dots \dots \dots$$

$$\frac{\partial y_n}{\partial t} + \sum_1^n \frac{\partial y_n}{\partial x_i} \frac{dx_i}{dt} - \frac{dy_n}{dt} = \frac{\partial y_n}{\partial t} + \frac{\partial H}{\partial x_n} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_n} = \lambda_n.$$

$$\frac{\partial y_{n+1}}{\partial t} + \sum_1^n \frac{\partial y_{n+1}}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial y_{n+1}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_{n+1}} = \mu_1,$$

$$\frac{\partial y_{n+2}}{\partial t} + \sum_1^n \frac{\partial y_{n+2}}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial y_{n+2}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_{n+2}} = \mu_2,$$

$$\dots \dots \dots$$

$$\frac{\partial y_{2n}}{\partial t} + \sum_1^n \frac{\partial y_{2n}}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial y_{2n}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i} \frac{\partial y_i}{\partial x_{2n}} = \mu_n.$$

Substituting $\lambda_i, \mu_i, a_{i,j}$ for the corresponding quantities in equations (1) and (2), the latter become

$$\left. \begin{aligned} \lambda_1 + \sum_1^n a_{1, n+i} \frac{dx_{n+i}}{dt} &= 0, \\ \lambda_2 + \sum_1^n a_{2, n+i} \frac{dx_{n+i}}{dt} &= 0, \\ &\dots \\ \lambda_n + \sum_1^n a_{n, n+i} \frac{dx_{n+i}}{dt} &= 0, \\ \mu_1 - \frac{dy_{n+1}}{dt} + \sum_1^n a_{n+1, n+i} \frac{dx_{n+i}}{dt} &= 0, \\ \mu_2 - \frac{dy_{n+2}}{dt} + \sum_1^n a_{n+2, n+i} \frac{dx_{n+i}}{dt} &= 0, \\ &\dots \\ \mu_n - \frac{dy_{2n}}{dt} + \sum_1^n a_{2n, n+i} \frac{dx_{n+i}}{dt} &= 0; \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \lambda_1 + \sum_1^n a_{1, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0, \\ \lambda_2 + \sum_1^n a_{2, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0, \\ &\dots \\ \lambda_n + \sum_1^n a_{n, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0, \\ \mu_1 + \frac{\partial H}{\partial x_{n+1}} + \sum_1^n a_{n+1, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0, \\ \mu_2 + \frac{\partial H}{\partial x_{n+2}} + \sum_1^n a_{n+2, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0, \\ &\dots \\ \mu_n + \frac{\partial H}{\partial x_{2n}} + \sum_1^n a_{2n, n+i} \frac{\partial H}{\partial y_{n+i}} &= 0. \end{aligned} \right\} \quad (4)$$

If in (3) we consider dx_{n+i}/dt and $-dy_{n+i}/dt$ ($i = 1, 2, \dots, n$), and in (4) $\partial H/\partial x_{n+i}$ and $\partial H/\partial y_{n+i}$ ($i = 1, 2, \dots, n$), as the unknown quantities, we shall have two systems of $2n$ equations of the first degree involving $2n$ unknown quantities. In the two systems the coefficients and the known terms are the

same ; hence the corresponding unknown quantities must be the same : whence

$$\frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial y_{n+i}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}. \quad (i = 1, 2, \dots, n) \quad (d)$$

Q. E. D.

2. It may be proved similarly that if

$$\frac{dx_{n+1}}{dt} + \frac{\partial H}{\partial y_{n+1}}, \quad \frac{dy_{n+1}}{dt} = -\frac{\partial H}{\partial x_{n+1}};$$

then

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},$$

and, in general, if any $2n$ of the $4n$ relations (c) and (d) exist, the remaining $2n$ relations will also hold true.

3. If H be expressed in terms of t and the x 's only, we shall have

$$\frac{dx_i}{dt} = 0; \quad (i = 1, 2, \dots, 2n)$$

and, by integration,

$$x_i = g_i,$$

in which the g 's are arbitrary constants ; whence

$$\frac{dy_i}{dt} = -\frac{\partial H}{\partial g_i}.$$

4. Similarly, if H be expressed in terms of t and the y 's only, we shall have

$$\frac{dy_i}{dt} = 0; \quad (i = 1, 2, \dots, 2n)$$

and, by integration,

$$y_i = g'_i,$$

in which the g 's are arbitrary constants ; whence

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial g'_i}.$$

5. If H be a function of $t, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ only, we shall have

$$\frac{dx_{n+i}}{dt} = 0, \quad \frac{dy_{n+i}}{dt} = 0; \quad (i = 1, 2, \dots, n)$$

and, by integration,

$$x_{n+i} = h_i, \quad y_{n+i} = h'_i, \quad (i = 1, 2, \dots, n)$$

in which the h 's and h' 's are arbitrary constants. In this case S may be stated to be a function of t and the n variables x_i and containing n arbitrary constants h_i . The equations

$$y_i = \frac{\partial S}{\partial x_i}, \quad h'_i = \frac{\partial S}{\partial h_i} \quad (i = 1, 2, \dots, n)$$

may then be defined as the general integrals of the simultaneous differential equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}. \quad (i = 1, 2, \dots, n)$$

6. If

$$H = (H - R) + R,$$

in which $H - R$ is a function of x_i 's and y_i 's only, while R is expressed in terms of x_{n+i} 's and y_{n+i} 's only; we shall have

$$\left. \begin{aligned} \frac{\partial H}{\partial x_i} &= \frac{\partial(H - R)}{\partial x_i}, & \frac{\partial H}{\partial y_i} &= \frac{\partial(H - R)}{\partial y_i}, \\ \frac{\partial H}{\partial x_{n+i}} &= \frac{\partial R}{\partial x_{n+i}}, & \frac{\partial H}{\partial y_{n+i}} &= \frac{\partial R}{\partial y_{n+i}}; \end{aligned} \right\} (i = 1, 2, \dots, n)$$

whence, if

$$\frac{dx_i}{dt} = \frac{\partial(H - R)}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial(H - R)}{\partial x_i},$$

then

$$\frac{dx_{n+i}}{dt} = \frac{\partial R}{\partial y_{n+i}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial R}{\partial x_{n+i}};$$

or, employing the idea of variation of constants,

$$\frac{dh_i}{dt} = \frac{\partial R}{\partial h'_i}, \quad \frac{dh'_i}{dt} = -\frac{\partial R}{\partial h_i}.$$

7. If we choose, the x 's and y 's may be divided into two groups containing $2n$ variables each, i. e. n x 's and n y 's, and by solving equations (b) the members of each group may be expressed as functions of the members of the other group only. Suppose such to be the case, and x_i, y_i to be any two members of one of the groups, and x_j, y_j to be any two members of the

remaining group. Then will

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}, \quad (a) \quad \frac{\partial x_i}{\partial y_j} = \frac{\partial x_j}{\partial y_i}, \quad (c)$$

$$\frac{\partial y_i}{\partial y_j} = -\frac{\partial x_j}{\partial x_i} \quad (b') \quad \frac{\partial x_i}{\partial x_j} = -\frac{\partial y_j}{\partial y_i}. \quad (d')$$

If i' and j' be particular values of i and j , we have from equations (b), remembering that variables belonging to the same group are independent of one another,

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial}{\partial x_{j'}} \frac{\partial S}{\partial x_{i'}} + \Sigma_i \left[\frac{\partial}{\partial x_i} \frac{\partial S}{\partial x_{i'}} \cdot \frac{\partial x_i}{\partial x_{j'}} \right], \quad (1')$$

$$\frac{\partial y_i}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}} \frac{\partial S}{\partial x_i} + \Sigma_j \left[\frac{\partial}{\partial x_j} \frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}} \right] = 0;$$

whence

$$\frac{\partial}{\partial x_i} \frac{\partial S}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}} \frac{\partial S}{\partial x_i} = -\Sigma_j \left[\frac{\partial}{\partial x_j} \frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}} \right]. \quad (2')$$

Substituting (2') in (1'), we have

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial}{\partial x_{j'}} \frac{\partial S}{\partial x_{i'}} - \Sigma_i \Sigma_j \left[\frac{\partial}{\partial x_j} \frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}} \cdot \frac{\partial x_i}{\partial x_{j'}} \right].$$

Similarly, interchanging i , i' and j , j' ,

$$\frac{\partial y_{j'}}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}} \frac{\partial S}{\partial x_{j'}} - \Sigma_j \Sigma_i \left[\frac{\partial}{\partial x_i} \frac{\partial S}{\partial x_j} \cdot \frac{\partial x_i}{\partial x_{j'}} \cdot \frac{\partial x_j}{\partial x_{i'}} \right];$$

whence

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial y_{j'}}{\partial x_{i'}}; \quad \text{or, in general,} \quad \frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}. \quad (a')$$

As may easily be seen,

$$\frac{\partial y_i}{\partial x_i} = \Sigma_j \left[\frac{\partial y_i}{\partial x_j} \frac{\partial x_j}{\partial x_i} + \frac{\partial y_i}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right] = 0, \quad (3')$$

$$\frac{\partial y_j}{\partial x_i} = \Sigma_i \left[\frac{\partial y_j}{\partial x_i} \frac{\partial x_i}{\partial x_j} + \frac{\partial y_j}{\partial y_i} \frac{\partial y_i}{\partial x_j} \right] = 0, \quad (4')$$

$$\frac{\partial x_i}{\partial y_i} = \Sigma_j \left[\frac{\partial x_i}{\partial y_j} \frac{\partial x_j}{\partial y_i} + \frac{\partial x_i}{\partial x_j} \frac{\partial y_j}{\partial y_i} \right] = 0, \quad (5')$$

$$\frac{\partial x_j}{\partial y_j} = \Sigma_i \left[\frac{\partial x_j}{\partial x_i} \frac{\partial x_i}{\partial y_j} + \frac{\partial x_j}{\partial y_i} \frac{\partial y_i}{\partial y_j} \right] = 0. \quad (6')$$

By reason of (a'), (3') and (4') may be written

$$\sum_j \frac{\partial y_i}{\partial x_j} \left[\frac{\partial x_j}{\partial x_i} + \frac{\partial y_j}{\partial y_i} \right] = 0,$$

$$\sum_i \frac{\partial y_i}{\partial x_j} \left[\frac{\partial x_i}{\partial x_j} + \frac{\partial y_i}{\partial y_j} \right] = 0;$$

whence (b') and (d') are derived directly by indeterminate coefficients. Also, by reason of (b') and (d'), (5') and (6') may be written

$$\sum_j \frac{\partial x_i}{\partial x_j} \left[\frac{\partial x_j}{\partial y_i} - \frac{\partial x_i}{\partial y_j} \right] = 0,$$

$$\sum_i \frac{\partial x_j}{\partial x_i} \left[\frac{\partial x_j}{\partial y_i} - \frac{\partial x_i}{\partial y_j} \right] = 0;$$

from either of which (c') follows immediately by indeterminate coefficients.

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